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Computation of a Degree of Controllability via System Discretization

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For the purpose of selecting actuator locations for control of systems such as large flexible spacecraft, a concept of the degree of controllability has recently been developed. The exact value of the degree of controllability cannot, in general, be determined in closed form. In this work, it is shown that a lower bound estimate of the exact degree of controllability can be easily determined by discretizing the continuous system to obtain an approximation to the recovery region. A computational procedure is developed to characterize this approximate recovery region and to determine the approximate degree of controllability as the minimum distance to the boundary of the region. Examples are presented to illustrate the relationship between step size and accuracy of the approximation.

Introduction

THE concept of controllability of a mathematical model of a physical system is one of the fundamental results of modern control theory and is an important tool in the design of control systems. As originally introduced by Kalman,[1] the check for complete controllability via the well-known rank test on the controllability matrix provides an answer to the *binary* question of whether or not the system is completely controllable. The relative merit of various completely controllable configurations of a system cannot be determined, however. It follows that a quantitative measure of controllability is needed. Such a result would be useful, for instance, when it is desired to evaluate various actuator locations for a distributed parameter system.

This problem was first addressed in an early study by Kalman, Ho, and Narendra,[2] who proposed a weighted trace of the controllability Gramian as a controllability measure. Brown[3] and Monzingo[4] addressed the question of observability of linear systems (and implied the treatment of controllability via duality). They recognized that the relationship between the column vectors of the observability matrix can identify the least observable direction in the state space. Johnson[5] extended the work of Kalman et al.,[2] considering in particular the trace and determinant of the inverse controllability Gramian as scalar measures of controllability. Müller and Weber[6] identified the maximum eigenvalue, trace, and determinant of the inverse controllability Gramian as three special cases of a more general form of scalar controllability measure dependent upon an independent parameter. Variation of that parameter leads to an infinite set of scalar measures of controllability, any of which can be used, for instance, in optimizing actuator placement.

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A common thread binds all of these developments, in that the proposed quantitative measures of controllability are based either on the controllability matrix or the controllability Gramian. Each leads to a measure which may be associated with the minimum energy or the average energy required to control the particular system. Optimization based on such a controllability measure therefore implicitly assumes the use of a minimum energy controller.

An entirely new approach was presented recently by Viswanathan, Longman, and Likins,[7,8] who proposed a scalar measure based on the set of recoverable states associated with a specified recovery time T. This degree of controllability is developed by considering the time optimal regulator problem. The results were extended by Longman and Alfriend[9] to handle the time optimal tracking problem.

Since the exact value of this degree of controllability is not available in general, a simple method was presented in Ref. 7 to obtain an approximation to the degree of controllability of any system; but this approximation is an upper bound giving an overly optimistic answer. For large flexible spacecraft shape control, with equations in modal form, the approximation can be shown to be tight, and periodically it is even exact. However, when it is desired to handle the system equations in their original form without diagonalizing, the lack of knowedge about the tightness of the bound under these circumstances necessitates the development of a new approximation technique.

The following section reviews the original development of the degree of controllability and its approximation as presented in Refs. 7 and 8. Subsequent sections present a new conservative approximation technique based on discretization of the continuous system. This is followed by a discussion of examples which illustrate the relation between the new approximation and the original approximation and investigate the behavior of the new approximation with respect to discretization step size.

Degree of Controllability

The system under consideration is that of the linear time-invariant regulator problem with bounded measurable controls. In terms of the normalized state vector x and the normalized control vector u, the system may be expressed as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$|u_i(t)| \leq 1 \qquad i = 1,2,\ldots,m \tag{2}$$

The control objective of a regulator control is to drive the state to $x=0$. A normalization of the state vector accomplishes the task of specifying the relative importance of driving each state element $x_i(t)$, $i=1,2,\ldots,n$, to zero.

For this system, the degree of controllability as developed by Viswanathan, Longman, and Likins[7] is based upon the following definitions.

Definition 1: The *recovery region for time T* for the normalized system, Eqs. (1) and (2), is the set

$$R = \{x(0) \mid \exists u(t), t\epsilon[0,T], |u_i(t)| \leq 1$$
$$\text{for} \quad i=1,2,\ldots m \quad \ni, x(T)=0\}$$

Definition 2: The degree of controllability in time T of the $x=0$ solution of the normalized system, Eqs. (1) and (2), is defined as

$$\rho = \inf \|x(0)\| \;\forall\; x(0) \notin R$$

where $\|\cdot\|$ represents the Euclidean norm.

The degree of controllability for time T is thus a scalar measure of the size of the recovery region, taken as the infimal distance from the origin to an initial state which cannot be returned to the origin in time T. Several remarks may be made regarding the implications of the above definitions.

1) Time optimal control of the system is implied since the boundary of the recovery region is the locus of all states which can be returned to the origin using minimum time control.

2) The previously mentioned normalization of the state will have an important influence on the degree of controllability since the norm of the state vector is dependent upon that normalization.

3) When the system Eq. (1) is uncontrollable in the binary sense, the recovery region for any time T will have dimension less than that of the state space, and therefore the degree of controllability will be zero.

The last remark is intuitively pleasing and was, in fact, established as a prerequisite for the development of the definition of degree of controllability.[7]

Upon introduction of this definition of degree of controllability, Viswanathan et al. subsequently addressed the problem of approximating the recovery region and obtained a general form for an approximate degree of controllability. The procedure involves choosing a set of n linearly independent directions in the state space and constructing a parallelepiped with sides parallel to these directions such that there is some point on each side which is in the recovery region, but such that no point outside the parallelepiped lies inside the recovery region. The minimum among the perpendicular distances to the sides of the parallelepiped is then taken as an approximation to the degree of controllability.

Any linearly independent spanning set (for example, the state space axes) will yield an approximating parallelepiped; however, the desired property that the approximate degree of controllability be zero if and only if the system is uncontrollable cannot be assured for an arbitrary choice of directions. This becomes evident when we consider a simple example.

In Fig. 1, the rectangle constructed with edges parallel to the state space axes provides a fairly tight approximation to recovery region 1. However, it yields a much poorer approximation to the degree of controllability ρ_2 of region 2. In fact, if $\rho_2 \to 0$ in such a way that region 2 degenerates to a line forming a diagonal of the rectangle, then the rectangle would yield a nonzero approximate degree of controllability $\bar{\rho}$ for a system which is, by definition, uncontrollable.

To preserve this fundamental characteristic of the degree of controllability, the above approach is abandoned in favor of an alternate set of (in general, nonorthogonal) directions. The real and imaginary parts of the eigenvectors and generalized eigenvectors of the A matrix are chosen as the n linearly independent directions in the state space. It has been shown (Refs. 7 and 8) that for these directions the desired property of the approximation can be demonstrated under fairly general assumptions. The choice of a nonorthogonal set of directions leads to a parallelepiped approximation to the recovery region as in Fig. 2.

The resulting approximate degree of controllability ρ^* is clearly an upper bound to the exact value. The accuracy of the approximation has been investigated for several simple systems,[10] but satisfactory behavior in general cannot be assured. In fact, in one case considered in Ref. 10, the approximation diverges from the exact value as the recovery time increases.

Discretization and the Discrete Recovery Region

The proposed approximation to the degree of controllability is obtained via discretization of the continuous system Eq. (1). This discretized version of a given system with piecewise constant controls leads to a recovery region in the state space which is contained within the recovery region of the original continuous system. The minimum distance to the boundary of this new recovery region provides a *lower* bound to the degree of controllability of the continuous system.

The standard approach to discretizing the system Eqs. (1) and (2) is developed by first writing the solution of Eq. (1) as

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau \tag{3}$$

If the total recovery time T is divided into N equal intervals ΔT and the control is restricted to be constant over each interval, then the state at the $(k+1)$ step, in terms of the state at the kth step is

$$x[(k+1)\Delta T] = e^{A\Delta T}x(k\Delta T) + e^{A\Delta T}\int_0^{\Delta T} e^{-A\tau}Bu(k\Delta T)\,d\tau \tag{4}$$

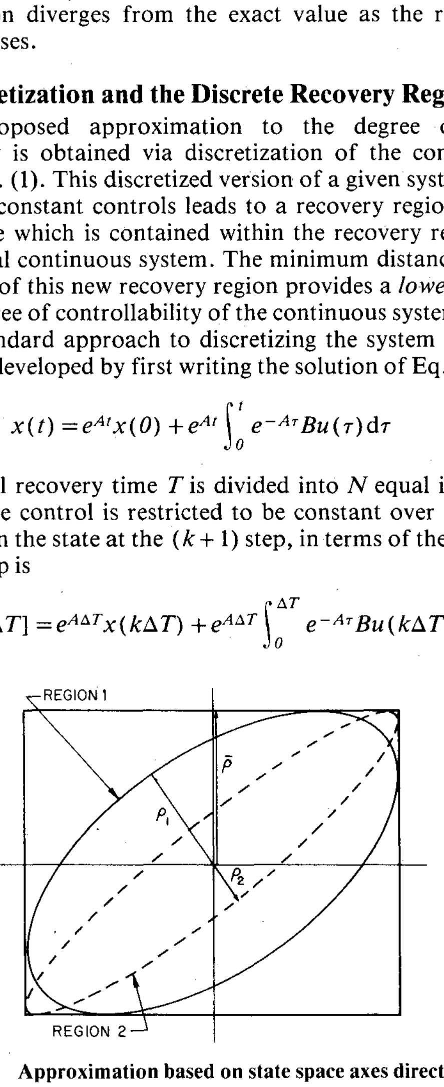

Fig. 1 Approximation based on state space axes directions.

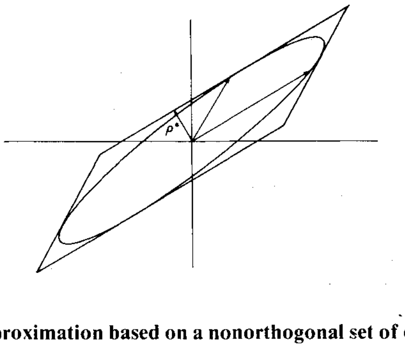

Fig. 2 Approximation based on a nonorthogonal set of directions.

or

$$x_{k+1} = e^{A\Delta T} x_k + \int_0^{\Delta T} e^{-A\lambda} B u_k \mathrm{d}\lambda \tag{5}$$

where

$$x_{k+1} \equiv x[(k+1)\Delta T] \tag{6}$$

$$u_k \equiv u(k\Delta T) \tag{7}$$

$$\lambda \equiv \Delta T - \tau \tag{8}$$

Defining the time-invariant matrices

$$G(\Delta T) = e^{A\Delta T} \tag{9}$$

$$H(\Delta T) = \left(\int_0^{\Delta T} e^{-A\lambda} \mathrm{d}\lambda\right) B \tag{10}$$

Eq. (5) becomes

$$x_{k+1} = Gx_k + Hu_k \tag{11}$$

An iterative substitution of Eq. (11) into itself yields an expression for the final state of the system in terms of the initial state $\bar{x}$ and the discrete control sequence:

$$x_N = G^N \bar{x} + \sum_{i=0}^{N-1} G^{N-1-i} H u_i \tag{12}$$

Restricting $x_N = 0$ in Eq. (12) will yield an expression for the set of all initial states $\bar{x} \epsilon \mathbb{R}^n$ which can be returned to the origin in N discrete steps.

$$[G^{N-1}H \mid G^{N-2}H \mid \ldots \mid H] \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = -G^N \bar{x} \tag{13}$$

This expression may be written more compactly as

$$\bar{x} = -G^{-N} F \mathbf{u} \tag{14}$$

where

$$F = [G^{N-1}H \mid G^{N-2}H \mid \ldots \mid H] \tag{15}$$

$$G^{-N} = (G^N)^{-1} \tag{16}$$

$$\mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \epsilon \mathbb{R}^{N \times m} \tag{17}$$

Imposing the control limitation Eq. (2) on each element of Eq. (17), Eq. (14) may be viewed as a mapping from the $N \times m$ dimensional control space (where m is the dimension of the control u_k at each step) to an n dimensional state space. Note that each point in the control space $\mathbb{R}^{N \times m}$ represents an entire control time history, not just an instantaneous value of the control as in the m dimensional control space of the continuous system. The set of admissible controls dictated by Eq. (2) forms a hypercube centered on the origin of the control space and bounded by orthogonal hyperplanes at unit distance from the origin. The state space recovery region for the discretized system is the image of this hypercube through the map $-G^{-N}F$. Generically, we must require $N \times m \geq n$ to assure that the recovery region is not dimensionally deficient. This relationship can be visualized (for $N=3$, $m=1$, $n=2$) in Fig. 3. It may be noted that the boundaries of the actual and approximate recovery regions need not contain points in common, although this is the case in the examples subsequently considered.

Several general characteristics of the resulting recovery region may be identified as a result of the convexity of the control hypercube and the linearity of the mapping:

1) The recovery region of the discretized system is convex.
2) It is bounded by hyperplanes.
3) Each hyperplane segment which constitutes a boundary of the recovery region is the image of an $n-1$ dimensional boundary segment of the control hypercube.

Note, however, that not every $n-1$ dimensional boundary segment of the control hypercube maps to a *boundary* segment of the recovery region. In Fig. 3, six edges of the cube form the recovery region boundary (solid lines) and six edges map to the interior of the recovery region (dashed lines).

A New Approximation

Based upon the preceding development, the following approximation to the degree of controllability is proposed.

Theorem 1: The minimum among the set of perpendicular distances to the hyperplanes of the discretized system recovery region (denoted ρ^*) is a conservative (lower bound) approximation to the degree of controllability of the continuous system.

The following two propositions establish the validity of the theorem.

Proposition 1: The approximate recovery region R^*, i.e., the recovery region of the discretized system, satisfies

$$R^* \subset R \tag{18}$$

Proof: This result follows directly from the fact that the set of admissible controls for the discrete system is a subset of the set of admissible controls for the continuous system ■

Proposition 1 establishes the discretized system recovery region as a conservative approximation to the continuous system recovery region. The next proposition establishes the proposed approximation as a lower bound to the continuous system degree of controllability.

Proposition 2: For $N \times m \geq n$, ρ^* of Theorem 1 satisfies

$$\rho^* \leq \rho \tag{19}$$

where ρ is defined by Definition 2.

Proof: To establish the above result, note first that the perpendicular distance from the origin to an arbitrary boundary hyperplane can be a distance to a point outside R. Such is the case in Fig. 3 for the hyperplanes (lines) which intersect the vertical axis. The fact that this cannot occur in the case of the hyperplane which determines ρ^*, is sufficient to prove the proposition, and will be shown by contradiction.

Assume that the minimum among all perpendicular distances, $d_{\min}$, is the distance to a point outside the discrete system recovery region. There must exist therefore another boundary hyperplane, which we denote j, such that the distance d to that hyperplane along the direction associated with $d_{\min}$ satisfies

$$d < d_{\min} \tag{20}$$

Since the minimum distance d_j to hyperplane j must satisfy

$$d_j \leq d \tag{21}$$

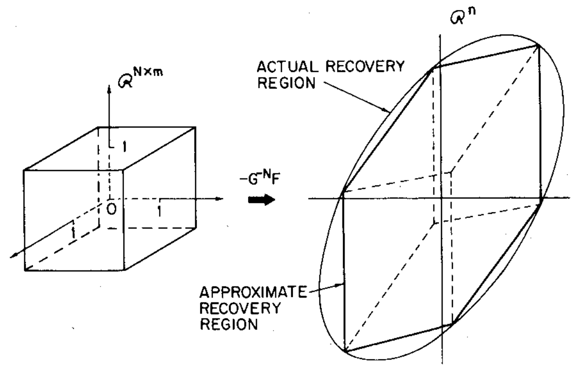

Fig. 3 Control space to state space mapping.

we have immediately

$$d_j < d_{\min} \tag{22}$$

which is a contradiction ■

Consider now the computation of the proposed approximation to the degree of controllability.

The linear transformation Eq. (14) from the discrete system control space to the state space may be rewritten

$$\bar{x} = K\mathbf{u} \tag{23}$$

where $K = -G^{-N}F$, and $\mathbf{u}$ is the entire control time history. We wish to determine the state space recovery region boundary associated with the set of admissible controls

$$|u_{ik}| \leq 1 \qquad i = 1,2,\ldots,m \qquad k = 1,2,\ldots,N \tag{24}$$

As noted earlier, the boundary of the recovery region, in general, comprises hyperplane segments associated with $n-1$ dimensional boundary segments of the control hypercube given by Eq. (24). However, not every boundary segment of the hypercube will map to a segment of the boundary of the recovery region. In fact, for any given set of parallel $n-1$ dimensional boundary segments of the control hypercube, only *two* will map to the boundary of the recovery region. [By virtue of Eq. (24) this pair will be symmetric about the origin.]

These ideas can be clarified by considering once more the specific case of Fig. 3. Here we wish to determine the *one-dimensional* boundary segments of the recovery region; therefore we choose to map the one-dimensional edges of the cube through the linear transformation $K = -G^{-N}F$. Under the action of the linear map, parallel lines in the control space are mapped to parallel lines in the state space. In this case, there are three distinct sets of edges; each set consisting of four parallel line segments. When each set is mapped into the state space, clearly, only two lines of each set will be extremal. This result will hold in general since we always wish to consider segments of dimension one less than that of the state space. The problem of determining an approximate degree of controllability is solved by 1) finding, for each set of parallel hyperplanes in the state space, the maximum perpendicular distance, and 2) among these maxima, choosing the minimum as the approximation.

The hyperplanes of a given set are generated by first partitioning the map in Eq. (23) as follows:

$$\bar{x} = [K_1 \ \ K_2]\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \tag{25}$$

where $\mathbf{u}_1 \epsilon \mathcal{R}^{n-1}$, $\mathbf{u}_2 \epsilon \mathcal{R}^{[(N\times m)-(n-1)]}$, and K_1 and K_2 are the corresponding partitions of K.

Each $n-1$ dimensional boundary of the hypercube can be characterized by fixing the values of $[N\times m - (n-1)]$ of the elements of u and allowing the remaining $(n-1)$ elements to vary between the limits of $+1$ and -1. Writing Eq. (25) more simply, as

$$\bar{x} = K_1\mathbf{u}_1 + K_2\mathbf{u}_2 \tag{26}$$

we note that for K_1 of maximal rank, there exists an n vector $\xi \neq 0$ such that

$$\xi^T K_1 = 0 \tag{27}$$

Multiplying Eq. (27) by ξ^T,

$$\xi^T \bar{x} = \xi^T K_2 \mathbf{u}_2 \tag{28}$$

Recalling that the elements of $\mathbf{u}_2$ are specified (either ± 1), we recognize Eq. (28) as the equation of a set of hyperplanes in the state space (parameterized by $\mathbf{u}_2$). The perpendicular distance to each hyperplane is given by

$$d = |\xi^T K_2 \mathbf{u}_2| / (\xi^T \xi)^{1/2} \tag{29}$$

where the signs of the elements of $\mathbf{u}_2$ are different for each hyperplane. If we let

$$z^T = \xi^T K_2 \tag{30}$$

and assume

$$|\xi| = 1 \tag{31}$$

then we recognize that the distance to the extremal member of the set of hyperplanes is

$$d_{\max} = \Sigma\, |z_i| \tag{32}$$

The computation of the approximate degree of controllability is reduced to determining ξ (typically via Gram-Schmidt orthonormalization) for each possible ordering of the columns in the partition of the matrix K and computing $d_{\max}$ according to Eq. (32). The number of $d_{\max}$ to be computed is equivalent to the number of ways of selecting $(n-1)$ columns from the $(N\times m)$ columns of K. The minimum among the set of all $d_{\max}$ is then the desired approximation.

Examples

Several considerations motivate the choice of the double integral plant as our first example. A typical model of a large flexible spacecraft, for which attitude and shape control is desired, will comprise both flexible and rigid body modes, when in modal form. The former take the form of harmonic oscillators, the latter double integral plants. In a separate work,[10] it was found that while the upper bound developed by Viswanathan et al. provides a tight approximation to the degree of controllability of harmonic systems (and indeed for particular periodic values of T, it is exact), in the case of the double integral plant, the approximation diverges from the exact value with increasing recovery time T. This results in a significant over-estimation of the degree of controllability. Additionally, this is one of the few simple examples in which the exact degree of controllability can be computed in closed form.

The continuous system model may be written

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{33}$$

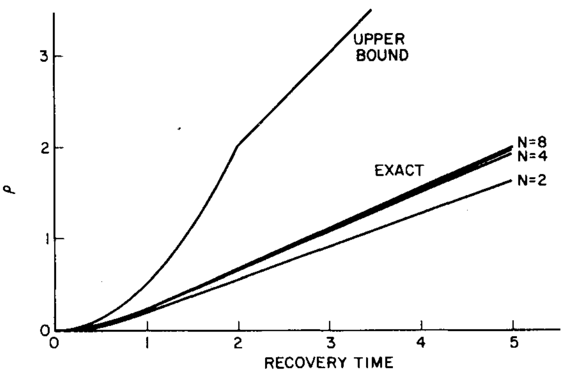

Fig. 4 Degree of controllability and approximations for double integral plant.

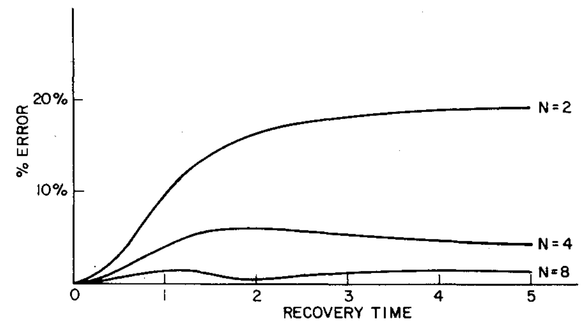

Fig. 5 Errors for conservative approximations.

In discrete form, the system is represented by Eq. (11) with

$$G=\begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} \tag{34}$$

and

$$H=\begin{bmatrix} (\Delta T)^2/2 \\ \Delta T \end{bmatrix} \tag{35}$$

where

$$\Delta T = T/N \tag{36}$$

We require $N \geq 2$ here since the state space is of dimension two and there is only one control input. Following the new conservative approximation technique put forth in the preceding section, the approximate degree of controllability may be computed for various values of discretization number N and recovery time T. Figure 4 presents the approximation for $N=2$, 4, 8 along with the exact value and the upper bound approximation. Readers are referred to Ref. 10 for the details of computation of the exact and upper bound values.

The accuracy of the approximation is evident from Fig. 5. For $N=8$, the error is never more than 3%. It is suggested that when N is to be increased to improve the accuracy of the approximation, it should be doubled at each step to assure an improved approximation. This is the logical approach when one considers that doubling N results in dividing each discrete control in half. The old set of admissible discrete controls is therefore a subset of the new set of admissible controls, guaranteeing that the approximation converges monotonically to the limit.

The second system considered is a simply supported beam. The model will include the first two modes, with actuators placed at one-third and two-thirds the length of the beam.

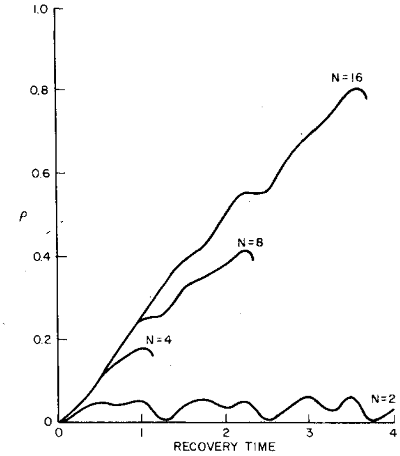

Fig. 6 Approximate values of degree of controllability for a simply supported beam.

The continuous system in normalized modal form is represented by the matrices

$$A=\begin{bmatrix} 0 & 2.47 & 0 & 0 \\ -2.47 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9.87 \\ 0 & 0 & -9.87 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} 0 & 0 \\ 0.866 & 0.866 \\ 0 & 0 \\ 0.216 & -0.216 \end{bmatrix} \tag{37}$$

The value of the approximate degree of controllability for the various values of N and T are illustrated in Fig. 6. For $N=2$, the approximation is valid over only a very short range of recovery time ($0<T<0.25$). Clearly, values of ρ for $T>0.5$ are meaningless since the degree of controllability of a linear time-invariant system must increase with increasing recovery time. Curves for $N=4$, 8, 16 are presented only over the range for which they are monotonically increasing.

It is not surprising that the approximation becomes zero for specific recovery times in the $N=2$ case. Recall that a discrete harmonic system is, by definition, uncontrollable whenever the step size is a multiple of the period of one of the modes. Since N is being held constant, the step size ΔT varies with recovery time T. The periods of the two modes considered here are 2.55 and 0.637. Therefore, when $T=1.274$ ($\Delta T=0.637$), the discrete system is uncontrollable and therefore the discrete approximation to the degree of controllability is zero.

In light of these results, it is recommended that in determining the approximate degree of controllability over a range of recovery time T, the step size ΔT should be fixed at a value smaller than the period of the fastest oscillator (rather than holding N constant). For a system for which periods are not known, this could be accomplished by first choosing the smallest allowable value of N and increasing the recovery time until the approximation is no longer increasing. Only a value of ΔT within the valid range for that N will lead to satisfactory behavior of the approximation.

Summary

A new conservative approximation technique has been developed for estimating the degree of controllability of general linear time-invariant systems. The procedure involves discretization of the continuous system and computation of the degree of controllability of the resulting discrete system. Computation of this value is reduced to performing a Gram-Schmidt orthonormalization on the columns of a linear mapping from the discrete control space to the state space.

The new approximation is shown to avoid the divergence problem associated with the original approximation technique. Discussion of a simple example leads to a straightforward approach to selecting the appropriate step size for the discretization.

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